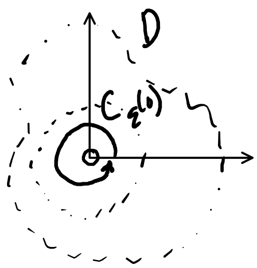
Example

(1) The function $f(z) = \frac{1}{z}$ has no antiderivative on $\mathbb{C}\setminus\{0\}$. In fact, it has no antiderivative on any domain containing a deleted neighborhood of 0. Take a circle $C_\varepsilon(0)$ so small that it lies in this domain. Then



$$\int_{C_\varepsilon} \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{\varepsilon e^{it}}\, \varepsilon i e^{it}\, dt$$

$$= i\int_0^{2\pi} 1\, dt = 2\pi i.$$

By FToCI, $f(z)$ has no antiderivative on such a domain. The problem is as follows: any branch $F(z) = \log z$ has derivative

$$F'(z) = \frac{1}{z}.$$

But $F(z)$ is not even defined on it branch cut. In fact, it is not possible to extend such a branch of log to an analytic function on all of $\mathbb{C}\setminus\{0\}$.

(2) $f(z) = \cos z$ is continuous on $\mathbb{C}$ and also $\sin z$ is entire. Moreover,

$$\frac{d}{dz}\sin z = \cos z = f(z).$$

So f has an antiderivative on $\mathbb{C}$. So for instance

$$\int_0^{\pi i} \cos z \, dz = \sin \pi i - \sin 0$$
$$= \sin \pi i.$$

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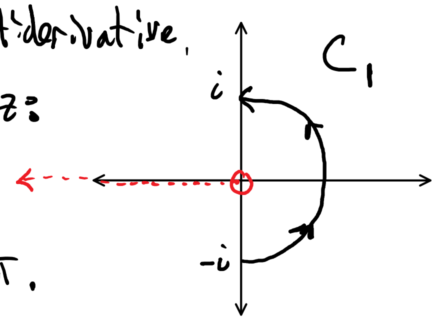
(3) Although $f(z) = \frac{1}{z}$ has no antiderivative on any domain containing 0, we can integrate f over a circle by using two different antiderivatives.

Let C_1 be parameterized by $z(t) = e^{it}$, $t \in [-\pi/2, \pi/2]$. On $\mathbb{C} \setminus (-\infty, 0]$, $f(z)$ has an antiderivative, namely the principal branch of $\log z$:

$$\operatorname{Log} z = \ln r + i\theta, \quad r > 0, \quad -\pi < \theta < \pi.$$

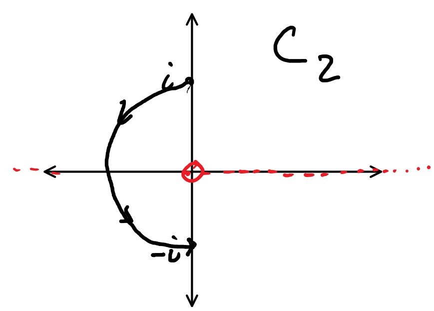


By the fundamental theorem,

$$\int_{C_1} \frac{1}{z} \, dz = \operatorname{Log} i - \operatorname{Log} -i$$
$$= \ln|i| + i\pi/2 - (\ln|-i| + i(-\pi/2))$$
$$= \pi i.$$

On the domain $\mathbb{C} \setminus [0, \infty)$, $f(z)$ has an antiderivative, namely

$$\log z = \ln r + i\theta, \quad r > 0, \quad 0 < \theta < 2\pi.$$



Hence, $$\int_{C_2} \frac{1}{z} \, dz = \log -i - \log i$$

$$= i\,3\pi/2 - i(\pi/2) = \pi i$$

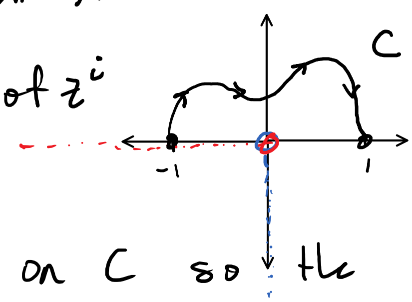
Hence, $$\int_C \frac{1}{z}\,dz = \int_{C_1} \frac{1}{z}\,dz + \int_{C_2} dz$$

$$= \pi i + \pi i = 2\pi i.$$

(4) Sometimes we can integrate a branch of a multiple valued function f even when the contour crosses the branch cut.

Let C be any curve joining -1 to 1 and lying above the x-axis (except the end points).

We will integrate the principal branch of z^i



$$f(z) = z^i = e^{i\,\mathrm{Log}\, z}$$

The function is piecewise continuous on C so the integral exists. The function $f(z)$ has an antiderivative on its domain of definition, but C does not live in that domain. But, the branch

$$g(z) = z^i = e^{i \log z}, \quad |z|>0 \quad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$$

The functions $f(z)$ and $g(z)$ agree everywhere on C (except at $z=-1$). But $g(z)$ has an antiderivative on a domain containing C, so

$$\int_C f(z)\,dz = \int_C g(z)\,dz$$

$$= \frac{1}{i+1}(1)^{i+1} - \frac{1}{i+1}(-1)^{i+1}$$

$$= \frac{1}{i+1}\left(1 - e^{-\pi}\right)$$

Cauchy-Goursat Theorem

The Cauchy-Goursat integral theorem gives a sufficient condition for the integral of a function over a simple closed curve to be zero. The theorem has powerful implications. Ultimately it leads to

- The Cauchy Integral formula.
- The theory of residues for computing contour integrals.
- A method to evaluate real-valued functions of a real variable, using contour integration.

Historically, the theorem was first proved by Cauchy with a weaker hypothesis. We prove this first.

formal symbol

Recall: (1) Contour integrals are related to line integrals. You can remember this by writing $f = u + iv$ and $dz = dx + idy$. Then

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C u\,dx - v\,dy + i\int_C u\,dy + v\,dx .$$

(2) Greens Theorem: suppose C is a simple closed contour in $\mathbb{R}^2$ and let R be the region enclosed by C. If P and Q have continuous first order partial derivatives on R. Then

$$\int_C P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA.$$

Theorem (Weak Cauchy Integral Theorem) Let C be a simple closed contour and let R denote the closed region consisting of C and its interior. If f is analytic on R and f' is continuous on R, then

$$\int_C f(z)\,dz = 0.$$

Proof. If f is analytic on R, then $u_x = v_y$ and $u_y = -v_x$ on R and

$$f' = u_x + i v_x$$
$$= v_y - i u_y.$$

Since f' is continuous, so are u_x, v_x, v_y, and u_y. Hence, by the above

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C u\,dx - v\,dy + i\int_C u\,dy + v\,dx.$$

(Green's Thm)

$$= \iint_R -v_x - u_y\,dA + i\int u_x - v_y\,dA$$

(Cauchy Riemann)

$$= \iint_R -V_x + V_x \, dA + i \iint_R U_x - U_x \, dA$$

$$= 0.$$

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Goursat was the first to prove that the continuity of f' can be omitted. This turns out to be essential for the theory of analytic functions. The problem is that it may be difficult to prove that the derivative of an analytic function is continuous!

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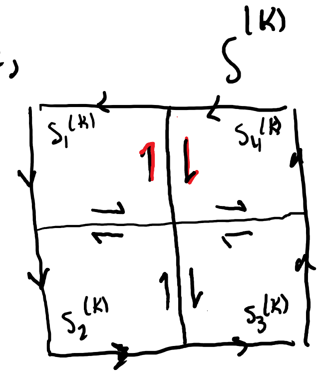
Theorem (Cauchy-Goursat Theorem) Let C be a simple closed contour and R the closed region consisting of C and its interior. If f is analytic on R, then

$$\int_C f(z)\,dz = 0.$$

Proof. (For simplicity, we will assume C is a square.)
The idea is to "divide and conquer". The idea is to break the curve into a finite number of smaller squares on which we can estimate the integral. We first construct a sequence of positively oriented curves $S^{(k)}$, each of which is the boundary of a square region $R^{(k)}$.
To begin, set $S^{(0)} \stackrel{\text{def}}{=} C$. Then, inductively, after the first k squares have been chosen, define the $(k+1)^{th}$ square as follows:

Divide $S^{(k)}$ into four congruent squares w/ positive orientation: $S_1^{(k)}, S_2^{(k)}, S_3^{(k)}, S_4^{(k)}$.

Notice that the integrals of f over the shared boundaries of these squares cancel. Hence,



$$\int_{S^{(k)}} f(z)\,dz = \sum_{i=1}^{4} \int_{S_i^{(k)}} f(z)\,dz.$$

Define
$$S^{(k+1)} \overset{\text{def}}{=} \max_{i=1}^{4} \left| \int_{S_i^{(k)}} f(z)\,dz \right|.$$

At this point, the sequence $S^{(0)}, \ldots, S^{(k)}, \ldots$ has been defined.

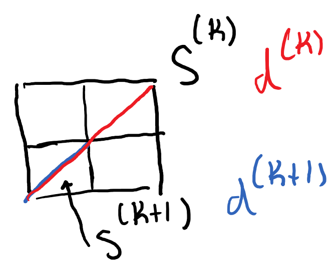
Notice that
$$\left| \int_{S^{(k)}} f(z)\,dz \right| \leq \sum_{i=1}^{4} \left| \int_{S_i^{(k)}} f(z)\,dz \right|$$
$$\leq 4 \left| \int_{S^{(k+1)}} f(z)\,dz \right|.$$

So by induction, for any $n \in \mathbb{N}$.

$$\boxed{(*) \quad \left| \int_{S^{(0)}=C} f(z)\,dz \right| \leq 4^n \left| \int_{S^{(n)}} f(z)\,dz \right|.}$$

Next, we record some more facts. Denote by $d^{(n)}$ the length of the diagonal of the nth square $S^{(n)}$ and denote by $p^{(n)}$ its perimeter.

Then
$$d^{(n)} = \frac{1}{2^n} d^{(0)}$$
$$p^{(n)} = \frac{1}{2^n} p^{(0)}.$$



Also, $d^{(n)} \longrightarrow 0$ as $n \longrightarrow \infty$.

Next, consider the associated sequence

$$R^{(0)} \supset R^{(1)} \supset \cdots \supset R^{(K)} \supset \cdots .$$

Each $R^{(K)}$ is compact (closed and bounded) and hence there is a unique point

$$z_0 \in \bigcap_{i=0}^{\infty} R^{(i)} .$$

A Fact assumed from analysis.

Since $z_0 \in R = R^{(0)}$, f is analytic at z_0. So we can define the following function:

$$\psi(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0) & , z \neq z_0 \\ 0 & , \text{otherwise}. \end{cases}$$

Then $\lim_{z \to z_0} \psi(z) = f'(z_0) - f'(z_0) = 0 = \psi(z_0)$ so ψ is continuous at z_0. We can write

$$f(z) = f(z_0) + \psi(z)(z - z_0) + f'(z_0)(z - z_0).$$

Note that $f(z_0)$, $f'(z_0)(z - z_0)$ have antiderivatives on $\mathbb{C}$. Hence, by the fundament theorem

$$\int_{S^{(n)}} f(z)\,dz = \cancelto{0}{\int_{S^{(n)}} f(z_0)\,dz} + \int_{S^{(n)}} \psi(z)(z - z_0)\,dz + \cancelto{0}{\int_{S^{(n)}} f'(z_0)(z - z_0)\,dz}$$

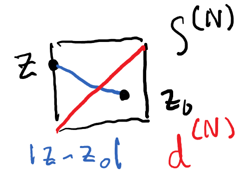
$$= \int_{S^{(n)}} \psi(z)(z - z_0)\,dz .$$

Let $\varepsilon > 0$. Since ψ is continuous at z_0, choose $\delta > 0$ such that

$$|z-z_0|<\delta \implies |\psi(z)|<\varepsilon.$$

Since $d^{(n)} \to 0$, choose $N \in \mathbb{N}$ such that $n \geq N$,

$|d^{(n)}| < \delta$. Hence, if $z \in S^{(N)}$, then $|z-z_0| < |d^{(N)}| < \delta$, and hence $|\psi(z)| < \varepsilon$.



Hence, we obtain

$$\left|\int_{S^{(N)}} f(z)\,dz\right| = \left|\int_{S^{(N)}} \psi(z)(z-z_0)\,dz\right|$$

$$\underset{\text{Triangle ineq}}{\leq} \max_{z\in S^{(N)}} |\psi(z)||z-z_0| \cdot \text{length } S^{(N)}.$$

$$\leq \varepsilon\, d^{(N)}\, p^{(N)}$$

$$= \varepsilon \cdot \frac{1}{2^N} \cdot \frac{1}{2^N}\, d^{(0)}\, p^{(0)}.$$

Hence, by (*)

$$\left|\int_{S^{(0)}=C} f(z)\,dz\right| \leq 4^N \left|\int_{S^{(N)}} f(z)\,dz\right|$$

$$\leq \varepsilon \cdot 4^n \frac{1}{2^n}\frac{1}{2^n}\, d^{(0)}\, p^{(0)} = \varepsilon\, d^{(0)} p^{(0)}.$$

The right hand side depends only on ε. Take $\varepsilon \to 0$ to obtain

$$\left|\int_C f(z)\,dz\right| = 0 \implies \boxed{\int_C f(z)\,dz.}$$

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